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Semifield planes of order q^4 with kernel F_{q^2} and center F_q

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Abstract

A classification of semifield planes of order q^4 with kernel F_{q^2} and center F_q is given. For q an odd prime, this proves the conjecture stated in [M. Cordero, R. Figueroa, On the semifield planes of order 5^4 and dimension 2 over the kernel, Note Mat. (in press)]. Also, we extend the classification of semifield planes lifted from Desarguesian planes of order q^2, q odd, obtained in [M. Cordero, R. Figueroa, On some new classes of semifield planes, Osaka J. Math. 30 (1993) 171–178], to the even characteristic case.

1. Introduction

Let $PG(3, q)$ be a three-dimensional projective space over F_q. A *spread* of $PG(3, q)$ is a set of q^2+1 mutually disjoint lines. Any spread $\mathcal{S}$ of $PG(3, q)$ defines via the construction of André/Bruck and Bose a translation plane $\pi(\mathcal{S})$ of order q^2 [4]. A *regulus* of $PG(3, q)$ is one ruling of a non-singular hyperbolic quadric $Q^+(3, q)$ of $PG(3, q)$. If l, m and n are

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three pairwise disjoint lines of $PG(3, q)$, there is a unique regulus $\mathcal{R}(l, m, n)$ of $PG(3, q)$ containing l, m and n. A spread $\mathcal{S}$ is said to be *regular* if $\mathcal{R}(l, m, n)$ is contained in $\mathcal{S}$ for all distinct lines l, m and n of $\mathcal{S}$. If $q > 2$, a spread $\mathcal{S}$ of $PG(3, q)$ is *regular* if and only if $\pi(\mathcal{S})$ is Desarguesian.

Let $\mathcal{S}$ be a spread of $PG(3, q)$ and choose the homogeneous projective coordinates (X_0, X_1, X_2, X_3) in $PG(3, q)$ in such a way that the lines $A = \{(0, 0, c, d) \mid c, d \in F_q\}$ and $B = \{(a, b, 0, 0) \mid a, b \in F_q\}$ belong to $\mathcal{S}$. For each line D of $\mathcal{S}$ different from A, there is a unique non-singular 2×2 matrix J_D over F_q such that $D = \{(a, b, c, d) \mid (c, d) = (a, b)J_D; a, b \in F_q\}$. The set $\mathcal{C}_{\mathcal{S}} = \{J_D \mid D \in \mathcal{S}\}$ has the following properties: (i) $\mathcal{C}_{\mathcal{S}}$ has q^2 elements, (ii) the zero matrix belongs to $\mathcal{C}_{\mathcal{S}}$, (iii) $X - Y$ is non-singular for all $X, Y \in \mathcal{C}_{\mathcal{S}}, X \neq Y$. Such a set $\mathcal{C}_{\mathcal{S}}$ is called the *spread set* associated with $\mathcal{S}$ with respect to A, B (see e.g. [9]). Conversely, starting from a set $\mathcal{C}$ of 2×2 matrices over F_q satisfying (i), (ii) and (iii), the set of lines $\mathcal{S} = \{l_M : M \in \mathcal{C}\} \cup \{A\}$ where $l_M = \{(a, b, c, d) : (c, d) = (a, b)M,\ a, b \in F_q\}$ is a spread of $PG(3, q)$ and $\mathcal{C}_{\mathcal{S}} = \mathcal{C}$.

A spread $\mathcal{S}$ is a *semifield spread* if $\mathcal{C}_{\mathcal{S}}$ is closed under sum (see [9]). If $\mathcal{S}$ is a non-regular semifield spread then the translation plane $\pi(\mathcal{S})$ is a *semifield plane* of order q^2 with *kernel* F_q (i.e., $\pi(\mathcal{S})$ is coordinatized by a *semifield* of dimension 2 over its kernel F_q).

Let $\pi(\mathcal{S})$ be a semifield plane defined by the semifield spread $\mathcal{S}$ of $PG(3, q)$, let $\mathcal{C}_{\mathcal{S}}$ be a spread set associated with $\mathcal{S}$ containing A and B, and let $V = V(4, q)$ denote the vector space of all 2×2 matrices over F_q. Since $\mathcal{C}_{\mathcal{S}}$ is closed under sum and contains the zero matrix, $\mathcal{C}_{\mathcal{S}}$ defines a vector subspace of V over some subfield of F_q. Denote by K the maximal subfield of F_q with respect to which $\mathcal{C}_{\mathcal{S}}$ is a K-vector subspace of V, i.e. K is the maximal subfield of F_q such that $\lambda X \in \mathcal{C}_{\mathcal{S}}$ for any $\lambda \in K$ and for any $X \in \mathcal{C}_{\mathcal{S}}$. If $\pi(\mathcal{S})$ is a non-Desarguesian semifield plane, then K is a proper subfield of F_q called the *center* of the semifield plane $\pi(\mathcal{S})$; equivalently, K is called the *center* of the semifield spread $\mathcal{S}$. It can be proven that if $(\mathcal{F}, +, \circ)$ is the semifield which coordinatizes the plane $\pi(\mathcal{S})$, then K is isomorphic to the center $\mathcal{K}$ of $\mathcal{F}$, where $\mathcal{K} = \{z \in N(\mathcal{F}) \mid z \circ d = d \circ z\, \forall d \in \mathcal{F}\}$, $N(\mathcal{F}) = N_l(\mathcal{F}) \cap N_m(\mathcal{F}) \cap N_r(\mathcal{F})$ is the *nucleus* of $\mathcal{F}$ and $N_l(\mathcal{F})$, $N_m(\mathcal{F})$, $N_r(\mathcal{F})$ are the *left, middle, right nucleus* of $\mathcal{F}$, respectively (see [3, p. 169 Definition 11.3.1]. The center of $\pi(\mathcal{S})$ has been characterized geometrically in [14] (see also [10]).

From now on suppose that $\pi(\mathcal{S})$ is a (non-Desarguesian) semifield plane and if $\pi(\mathcal{S})$ has center K, refer to it as a K*-semifield plane* and to $\mathcal{S}$ as a K*-semifield spread*.

Let $\pi(\mathcal{S})$ be a K-semifield plane and let $K = F_s$ $(q = s^n)$. Since $\mathcal{C}_{\mathcal{S}}$ is an F_s-vector subspace of V of dimension $2n$, it defines in $\mathbb{P} = PG(V, F_q)$ an F_s*-linear subset* of rank $2n$, namely $L(\mathcal{C}_{\mathcal{S}}) = \{\langle M \rangle_{F_q} : M \in \mathcal{C}_{\mathcal{S}} \setminus \{0\}\}$. The matrices defining $L(\mathcal{C}_{\mathcal{S}})$ are non-singular; hence the linear set $L(\mathcal{C}_{\mathcal{S}})$ is disjoint from the hyperbolic quadric $Q = Q^+(3, q)$ of $\mathbb{P}$ with equation $X_0X_3 - X_1X_2 = 0$. To the linear set $L(\mathcal{C}_{\mathcal{S}})$ associated with $\pi(\mathcal{S})$ there corresponds the linear set associated with the translation ovoid of the hyperbolic quadric $Q^+(5, q)$ defined by $\mathcal{S}$ via the Plücker map; these linear sets have been studied by Lunardon in [15].

In this paper, we first remark that two semifield planes $\pi(\mathcal{S}_1)$ and $\pi(\mathcal{S}_2)$ with center F_s are isomorphic if and only if the associated F_s-linear sets are collinear under the action of the subgroup G of $P\Gamma O^+(4, q)$ fixing the reguli of the hyperbolic quadric Q. Then we focus on the case $n = 2$, and we prove a classification theorem for F_s-linear sets of rank 4 in $PG(3, s^2)$ disjoint from $Q^+(3, s^2)$, up to the action of the group G. As a consequence of this result, relying on the above mentioned connection between semifield planes and linear sets, we give a classification of semifield planes of order s^4, with kernel F_{s^2} and center F_s.

That is, if s is odd, we prove that a semifield plane π of order s^4, with kernel F_{s^2} and center F_s, belongs to one of the following classes: generalized Dickson semifield planes, Hughes–Kleinfeld semifield planes, semifield planes lifted from Desarguesian planes of order s^2 of Cordero–Figueroa type, generalized twisted field planes. In the case where s is an odd prime, this classification result proves the conjecture stated by Cordero and Figueroa in [8]. Furthermore, we prove that if s is even, a semifield plane π of order s^4, with kernel F_{s^2} and center F_s either is one of Hughes–Kleinfeld semifield plane or a generalized twisted field plane or belongs to a class of semifield planes not isomorphic to the previous ones. The planes of this latter class can be also obtained by lifting a Desarguesian plane of order s^2, s even. As a consequence of our results we obtain a classification of semifield planes of order p^4, p prime, with kernel containing F_{p^2}, and we extend the classification of semifield planes lifted from Desarguesian planes of order q^2, q odd, obtained by Cordero and Figueroa in [7], to the even characteristic case.

2. F_s-semifield planes and F_s-linear sets

Let $q = s^n$. As pointed out in Section 1, if $\pi(\mathcal{S})$ is an F_s-semifield plane associated with an F_s-semifield spread $\mathcal{S}$ of $\mathbb{P} = PG(3, q)$ and $\mathcal{C}_\mathcal{S}$ is a spread set associated with $\mathcal{S}$, then $\mathcal{C}_\mathcal{S}$ defines an F_s-linear set $L(\mathcal{C}_\mathcal{S})$ of $\mathbb{P}$ of rank $2n$ disjoint from the hyperbolic quadric Q with equation $X_0X_3 - X_1X_2 = 0$. Conversely, let L be an F_s-linear set of $\mathbb{P}$ of rank $2n$, $n > 1$, with F_s as the maximal subfield of linearity and assume that L is disjoint from Q. If $\mathcal{C}$ is an F_s-linear subspace of V defining L, then $\mathcal{C}$ has dimension $2n$ over F_s, i.e. $|\mathcal{C}| = s^{2n} = q^2$, $\mathcal{C}$ is closed under sum and moreover, since $L \cap Q = \emptyset$, each matrix of $\mathcal{C}$ is non-singular. This means that $\mathcal{C}$ is a spread set of a non-regular semifield spread $\mathcal{S}$ of $PG(3, s^n)$ with center F_s, i.e. L is the linear set associated with the semifield plane $\pi(\mathcal{S})$.

Let $\pi(\mathcal{S}_1)$ and $\pi(\mathcal{S}_2)$ be two F_s-semifield planes and let $\mathcal{C}_{\mathcal{S}_1}$ and $\mathcal{C}_{\mathcal{S}_2}$ be the associated spread sets. The planes $\pi(\mathcal{S}_1)$ and $\pi(\mathcal{S}_2)$ are isomorphic if and only if there exists a map $\phi: \mathcal{C}_{\mathcal{S}_1} \to \mathcal{C}_{\mathcal{S}_2}$, such that $\phi(M) = AM^\sigma B$, where A and B are non-singular 2×2 matrices over F_{s^n} and $\sigma \in \mathrm{Aut}(F_{s^n})$ [16]. In this case we say that the spread sets $\mathcal{C}_{\mathcal{S}_1}$ and $\mathcal{C}_{\mathcal{S}_2}$ are *equivalent*.

Theorem 2.1. *The planes $\pi(\mathcal{S}_1)$ and $\pi(\mathcal{S}_2)$ are isomorphic if and only if $L(\mathcal{C}_{\mathcal{S}_1})$ and $L(\mathcal{C}_{\mathcal{S}_2})$ are collinear with respect to the subgroup G of $P\Gamma O^+(4, q)$ fixing the reguli of the hyperbolic quadric $Q^+(3, q)$.*

Proof. Suppose that $\pi(\mathcal{S}_1)$ and $\pi(\mathcal{S}_2)$ are isomorphic and let $\phi : \mathcal{C}_{\mathcal{S}_1} \to \mathcal{C}_{\mathcal{S}_2}$, $\phi(M) = AM^\sigma B$, with A and B non-singular 2×2 matrices over F_q. Then ϕ induces a semilinear collineation $\overline{\phi}$ of $\mathbb{P} = PG(3, q)$ (the three-dimensional projective space defined by the four-dimensional vector space of the 2×2 matrices over F_q), with associated automorphism σ, whose linear part is given by the 4×4 matrix over F_q:

$$\begin{pmatrix} a_1b_1 & a_1b_3 & a_2b_1 & a_2b_3 \\ a_1b_2 & a_1b_4 & a_2b_2 & a_2b_4 \\ a_3b_1 & a_3b_3 & a_4b_1 & a_4b_3 \\ a_3b_2 & a_3b_4 & a_4b_2 & a_4b_4 \end{pmatrix}, \tag{*}$$

with $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$. Hence $\overline{\phi}$ is an element of the subgroup G of $P\Gamma O^+(4, q)$ fixing the reguli of the hyperbolic quadric $Q^+(3, q)$ (see, e.g., [12]) and, since $\phi(M) = AM^\sigma B \in \mathcal{C}_{\mathcal{S}_2}$ for any $M \in \mathcal{C}_{\mathcal{S}_1}$, $\overline{\phi}(L(\mathcal{C}_{\mathcal{S}_1})) = L(\mathcal{C}_{\mathcal{S}_2})$.

Conversely, suppose that $g(L(\mathcal{C}_{\mathcal{S}_1})) = L(\mathcal{C}_{\mathcal{S}_2})$, with $g \in G$, and let σ be the automorphism associated with g. Then by [12, p. 28], the matrix associated with g is given by (*); if we put $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ and $\phi(M) = AM^\sigma B$, then $\phi(\mathcal{C}_{\mathcal{S}_1}) = \mathcal{C}_{\mathcal{S}_2}$. Hence $\pi(\mathcal{S}_1)$ and $\pi(\mathcal{S}_2)$ are isomorphic. □

By Theorem 2.1, the study of F_s-semifield planes two-dimensional over their kernel and four-dimensional over their center is equivalent to the study of F_s-linear sets of rank $2n$ of $\mathbb{P}$ (with F_s as the maximal subfield of linearity) disjoint from the hyperbolic quadric Q, under the action of G.

In the next section, we focus on the case $n = 2$ studying F_q-linear sets of $PG(3, q^2)$, disjoint from $Q^+(3, q^2)$, under the action of G.

3. F_q-linear sets of $PG(3, q^2)$

Let $\mathbb{P} = PG(3, q^2)$ be a three-dimensional projective space over F_{q^2} and denote by $V = V(4, q^2)$ the vector space associated with $\mathbb{P}$. Let $Q = Q^+(3, q^2)$ be the hyperbolic quadric of $\mathbb{P}$ with equation $X_0X_3 - X_2X_1 = 0$. If $\mathcal{R}$ is one ruling of Q, denote by $\mathcal{R}'$ the opposite ruling. Also, let L be an F_q-linear set of $\mathbb{P}$ of rank 4 disjoint from Q having F_q as the maximal subfield of linearity and let U be an F_q-vector subspace whose vectors define the points of L.

Let $P = \langle u \rangle_{F_{q^2}}, u \in U$, be a point of L; we say that P has *weight* i in L if $\dim_{F_q}(U \cap \langle u \rangle_{F_{q^2}}) = i$; in our case $i = 1, 2$. If there are at least two points, say P and Q, in L of weight 2 then L is a line of $PG(3, q^2)$, i.e. U is a two-dimensional vector space over F_{q^2} and F_q is not the maximal subfield of linearity of L, a contradiction.

If L contains a point P_0 of weight 2, then L is contained in a plane π of $\mathbb{P}$ and since $L \cap Q = \emptyset$, π is a non-tangent plane to Q. As G acts transitively on the set of non-tangent planes to Q, we can suppose, without loss of generality, that π has equation $X_3 = X_0$. It is easy to see that L is an F_q-linear set of π of rank 4, a union of lines through the point P_0 and disjoint from the irreducible conic Γ: $X_0^2 - X_1X_2 = 0$ of π. Such a linear set is associated with a non-linear *semifield flock* of the quadratic cone in which all the planes pass through a common point. By [17], such a set exists only if q is odd and P_0 is a point exterior to Γ. Since the stabilizer G_π of the plane π in the group G acts transitively on the set of points external to Γ, we can suppose $P_0 = (1, 0, 0, 1)$. As P_0 has weight 2, we can write $L = \{(x, f(y), g(y), x) : x, y \in F_{q^2}\}$, where $f, g : F_{q^2} \to F_{q^2}$ are linear functions over F_q and $x^2 \neq f(y)g(y), \forall (x, y) \in F_{q^2} \times F_{q^2} \setminus \{(0, 0)\}$. Since $f(y)$ and $g(y)$ are permutation polynomials, we can suppose $f(y) = y$; so $\forall (x, y) \neq (0, 0)$, $x^2 \neq yg(y)$, i.e. $yg(y)$ is a non-square of $F_{q^2} \forall y \neq 0$. By [6], we get $g(y) = my^q$, where m is a non-square of F_{q^2}. Hence, by [17], we have proved the following:

Proposition 3.1. *Up to the action of G, if L is an F_q-linear set of rank 4 disjoint from Q with exactly one point of weight 2, then q is odd and L is*

$$L = \{(x, y, my^q, x) : (x, y) \in F_{q^2} \times F_{q^2} \setminus (0, 0)\},$$

where m is a non-square in F_{q^2}. $\square$

Now let us consider the case where all the points of L have weight 1. Then $\langle U \rangle_{F_{q^2}} = V$ and any basis of U over F_q is also a basis of V over F_{q^2}; i.e., if $\{u_0, u_1, u_2, u_3\}$ is an F_q-basis of U, then $L = \{\langle \alpha_0 u_0 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 \rangle : \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in F_q\} \simeq PG(3, q)$. This means that L is a canonical subgeometry of $PG(3, q^2)$. Each canonical subgeometry Σ of $PG(3, q^2)$ can be obtained as the set of fixed points of a semilinear collineation τ of $PG(3, q^2)$ of order 2; i.e. $\Sigma = \mathrm{Fix}(\tau)$, $\tau : (x_0, x_1, x_2, x_3) \to A(x_0^q, x_1^q, x_2^q, x_3^q)$, $A \in GL(4, q^2)$ and $A^q A = \rho I$, $\rho \in F^*_{q^2}$.

Lemma 3.2. *Let Σ be a canonical subgeometry of $PG(3, q^2)$ disjoint from a hyperbolic quadric $Q = Q^+(3, q^2)$, and let $\tau \in P\Gamma L(4, q^2)$ be the semilinear collineation of $PG(3, q^2)$ of order 2 such that $\Sigma = \mathrm{Fix}(\tau)$. Then $Q \cap Q^\tau$ is one of the following configurations:*

(a) *$Q \cap Q^\tau = \{l, l^\tau, m, m^\tau\}$, where $l, l^\tau \in \mathcal{R}$ and $m, m^\tau \in \mathcal{R}'$;*

(b) *$Q \cap Q^\tau = \{C, C^\tau\}$, where C is a non-degenerate conic such that $C \cap C^\tau = \{P, P^\tau\}$, for some point P;*

(c) *$Q \cap Q^\tau = \emptyset$ in $PG(3, q^2)$ and $Q^* \cap Q^{*\tau^*} = \{l, l^{\tau^*}, l^{\tau^{*2}}, l^{\tau^{*3}}\}$, with l and $l^{\tau^{*2}} \in \mathcal{R}^*$ and $l^{\tau^*}, l^{\tau^{*3}} \in \mathcal{R}'^*$, where Q^*, τ^*, $\mathcal{R}^*$ and $\mathcal{R}'^*$ are the extensions of Q, τ, $\mathcal{R}$ and $\mathcal{R}'$ to $\mathbb{P}^* = PG(3, q^4)$, respectively.*

Proof. Since any two canonical subgeometries of $PG(3, q^2)$ are isomorphic, we can suppose that $\Sigma = \mathrm{Fix}(\tau)$ with $\tau : (X_0, X_1, X_2, X_3) \to (X_0^q, X_1^q, X_2^q, X_3^q)$, i.e. $\Sigma = \{(c_0, c_1, c_2, c_3) : c_i \in F_q\}$. Let $b(X_0, X_1, X_2, X_3) = \sum_{i,j=0 i \le j}^{3} a_{ij} X_i X_j = 0$ be the equation of Q and choose an element $\xi \in F_{q^2} \setminus F_q$. Since $\{1, \xi\}$ is an F_q-basis of F_{q^2}, we can write $a_{ij} = \alpha_{ij} + \xi \beta_{ij}$ with $\alpha_{ij}, \beta_{ij} \in F_q$, and hence $b(X_0, X_1, X_2, X_3) = \sum \alpha_{ij} X_i X_j + \xi \sum \beta_{ij} X_i X_j$.

Since Σ is disjoint from Q, the equations $\sum_{i,j=0 i \le j}^{3} \alpha_{ij} X_i X_j = 0$ and $\sum_{i,j=0 i \le j}^{3} \beta_{ij} X_i X_j = 0$ determine a pencil $\mathcal{P}$ of quadrics of Σ with empty intersection. Note that the two quadrics Q and Q^τ (over F_{q^2}) belong to the pencil $\mathcal{P}$. In [5, Theorem 5.1], all the possibilities for the base curve of a pencil of quadrics in a projective space have been classified; in the case of empty intersection the base curve is necessarily reducible and the possibilities are the following: $Q \cap Q^\tau = \{l, l^\tau, m, m^\tau\}$, where $l, l^\tau \in \mathcal{R}$ and $m, m^\tau \in \mathcal{R}'$; $Q \cap Q^\tau = \{C, C^\tau\}$, where C is a non-degenerate conic of Q or $Q \cap Q^\tau = \emptyset$

in $PG(3,q^2)$ and $Q^* \cap Q^{*\tau^*} = \{l, l^{\tau^*}, l^{\tau^{*2}}, l^{\tau^{*3}}\}$ in $PG(3,q^4)$, where Q^* and τ^* are the extensions of Q and τ to $\mathbb{P}^* = PG(3,q^4)$, respectively. Note that, if $Q \cap Q^\tau = \{C, C^\tau\}$ then $C \cap C^\tau = \{P, P^\tau\}$ for some point P. Indeed, let $\pi = \langle C\rangle$ and $\pi^\tau = \langle C^\tau\rangle$ be the planes of $\mathbb{P}$ containing C and C^τ respectively, and let $m = \pi \cap \pi^\tau$. Denote by m^* the quadratic extension of m over F_{q^4} and let $P \in m^* \cap Q^*$. Let C^* be the extension of the conic C over F_{q^4} and note that the point P belongs to $C^* \cap C^{*\tau}$. Since τ^{*2} acts as the identity on the subconic C of C^*, then τ^{*2} fixes the conic C^*. So the points P^{τ^*}, $P^{\tau^{*2}}$ and $P^{\tau^{*3}}$ also belong to $C^* \cap C^{*\tau}$. Since $\Sigma \cap C = \emptyset$, we get $P = P^{\tau^{*2}}$ and $P^{\tau^*} = P^{\tau^{*3}}$, i.e. P and $P^\tau = P^{\tau^*}$ belong to $m \cap Q$ and hence $C \cap C^\tau = \{P, P^\tau\}$. □

We say that a subgeometry $\Sigma = \mathrm{Fix}(\tau)$ is *of type* (a), (b) or (c) if the geometric configuration $Q \cap Q^\tau$ is as in the case (a), (b) or (c), respectively.

Remark 3.3. If two canonical subgeometries $\Sigma = \mathrm{Fix}(\tau)$ and $\Sigma' = \mathrm{Fix}(\tau')$ are collinear under the action of the group G, then they are of the same type. Indeed, if g is an element of G such that $\Sigma^g = \Sigma'$, then $\tau' = g^{-1}\tau g$ and hence $Q \cap Q^{\tau'} = Q \cap Q^{g^{-1}\tau g} = (Q \cap Q^\tau)^g$. This means that if Σ and Σ' are of different types, then they do not belong to the same orbit under the group action G.

The main result of this section is the classification of the canonical subgeometries of $PG(3,q^2)$ disjoint from the hyperbolic quadric $Q = Q^+(3,q^2)$, under the group action G. To this end we need some properties of the action of the group G.

In what follows, the linear part of G is denoted by $\overline{G}$, the reguli of Q are $\mathcal{R}'$ and $\mathcal{R}$, with $\mathcal{R}' = \{l'_{\lambda,\mu}: \lambda, \mu \in F_{q^2}\}$, $l'_{\lambda,\mu} = \{(\lambda X_0, \mu X_0, \lambda X_2, \mu X_2): X_0, X_2 \in F_{q^2}\}$ and $\mathcal{R} = \{l_{\lambda,\mu}: \lambda, \mu \in F_{q^2}\}$, $l_{\lambda,\mu} = \{(\lambda X_0, \lambda X_1, \mu X_0, \mu X_1): X_0, X_1 \in F_{q^2}\}$. An element ω of $\overline{G}$ is defined by a matrix $A = B'B$, where

$$B' = \begin{pmatrix} a' & c' & 0 & 0 \\ b' & d' & 0 & 0 \\ 0 & 0 & a' & c' \\ 0 & 0 & b' & d' \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & c \\ b & 0 & d & 0 \\ 0 & b & 0 & d \end{pmatrix}, \tag{1}$$

with $a'd' - c'b' \neq 0$ and $ad - cb \neq 0$, i.e. $\omega = \theta'\theta$ where θ' and θ are the linear collineations of G defined by the matrices B' and B respectively. The collineations of type θ' (respectively θ) of $\overline{G}$ fix the regulus $\mathcal{R}$ (respectively $\mathcal{R}'$) linewise and they form a subgroup of $\overline{G}$ acting on the regulus $\mathcal{R}'$ (respectively $\mathcal{R}$) as the linear group $PGL(2,q^2)$.

Proposition 3.4. *Let m be a line of $PG(3,q^2)$ not contained in Q and denote by $m^\perp$ the polar line of m with respect to the orthogonal polarity defined by Q. Let $\overline{G}_m$ be the stabilizer in $\overline{G}$ of m. Then the following hold:*

(i) *If m is a non-tangent line to Q then $\overline{G}_m$ acts transitively on $m \setminus \{m \cap Q\}$.*

(ii) *If m is a line external to Q and M is any point on m, then $\overline{G}_{m,M}$ (the stabilizer of M in $\overline{G}_m$) has two orbits on $m^\perp$ if q is odd, while $\overline{G}_{m,M}$ acts transitively on $m^\perp$ if q is even.*

Proof. First suppose that m is a secant line to Q. Since G acts transitively on the secants we can suppose that $m = \{(X_0, 0, 0, X_3): X_0, X_3 \in F_{q^2}\}$. In this case, the stabilizer of m in $\overline{G}$ consists of the matrices

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & v & 0 \\ 0 & u & 0 & 0 \\ uv & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & uv \end{pmatrix}$$

with $u, v \in F_{q^2} \setminus \{0\}$. It is straightforward to see that $\overline{G}_m$ acts transitively on $m \setminus \{m \cap Q\}$.

Now, let q be odd and suppose m is a line external to Q. Since the group G acts transitively on the external lines, we can suppose that $m = \{(X_0, X_1, \sigma X_1, X_0): X_0, X_1 \in F_{q^2}\}$, where σ is a non-square in F_{q^2}, and hence $m^\perp = \{(X_0, X_1, -\sigma X_1, -X_0): X_0, X_1 \in F_{q^2}\}$.

If ϕ is an element of F_{q^4} such that $\phi^2 = \sigma$, then $l'_{\pm\phi,1}$ and $l_{1,\pm\phi}$ are the lines of the quadric Q^* intersecting m on $PG(3, q^4)$. If ω is an element of $\overline{G}_m$, then either $\omega(l'_{\phi,1}) = l'_{\phi,1}$ and $\omega(l_{1,\phi}) = l_{1,\phi}$ or $\omega(l'_{\phi,1}) = l'_{-\phi,1}$ and $\omega(l_{1,\phi}) = l_{1,-\phi}$. In the first case, the matrix A associated with ω has to satisfy the conditions $a = d$, $b = c\sigma$, $a' = d'$ and $b' = c'\sigma^{-1}$. In the second case we have $a = -d$, $b = -c\sigma$, $a' = -d'$ and $b' = -c'\sigma^{-1}$. Therefore, it can be easily shown that $\overline{G}_m$ acts transitively on the points of m. This implies that we can fix $M = (1, 0, 0, 1) \in m$. With this choice for M, an element of $\overline{G}_m$ belongs to $\overline{G}_{m,M}$ if and only if $c'a = -\sigma a'c$. Fix the point $R = (1, 0, 0, -1)$ on $m^\perp$ and denote by $O(R)$ the orbit of R under the action of $\overline{G}_{m,M}$. We have $O(R) = \{(\frac{a^2+c^2\sigma}{2ac}, 1, -\sigma, -\frac{a^2+c^2\sigma}{2ac}): a, c \in F^*_{q^2}\} \cup \{R\}$. Hence, the point $(t, 1, -\sigma, -t) \in m^\perp$ belongs to $O(R)$ if and only if $t^2 - \sigma$ is a square in F_{q^2}. This implies that there are two orbits of points on $m^\perp$ under the action of $\overline{G}_{m,M}$: the orbit $O(R)$ and the orbit $O(T)$ where $T = (0, 1, -\sigma, 0)$.

Suppose that q is even and let σ be an element of F_{q^2} such that the polynomial $x^2 + x + \sigma = 0$ is irreducible over F_{q^2}, i.e. $\mathrm{Tr}_{q^2/q}(\sigma) = 1$. Since the group G acts transitively on the set of lines external to Q, we can suppose $m = \{(X_0, X_1, \sigma X_1, X_0 + X_1): X_0, X_1 \in F_{q^2}\}$, so $m^\perp = \{(X_0, X_1, X_0 + \sigma X_1, X_0): X_0, X_1 \in F_{q^2}\}$. With the same arguments as in the case where q is odd, we obtain that the elements of the stabilizer $\overline{G}_m$ are defined (see Eq. (1)) by matrices $A = B'B$ such that $a' = d' + (1-\delta)b'$, $c' = b'\sigma + \delta d'$, $b = c\sigma + \delta a$, $d = a + (1-\delta)c$, where $\delta = 0, 1$ and $(a, c), (b', d') \neq (0, 0)$. Also, in this case, an easy computation shows that $\overline{G}_m$ acts transitively on m. Hence we can fix the point $M = (1, 0, 0, 1) \in m$, and, in this case, an element of $\overline{G}_m$ belongs to $\overline{G}_{m,M}$ if $ab' + cd' = 0$. Fix the point $R = (0, 1, \sigma, 0) \in m^\perp$ and let $O(R)$ be the orbit of R under the group action $\overline{G}_{m,M}$. The points $R^\omega \in O(R)$, where $\omega \in \overline{G}_{m,M}$ with $\delta = 0$, are $\left\{\left(1, \frac{a^2+(c\sigma)^2}{c^2\sigma}, 1 + \sigma\left(\frac{a^2+(c\sigma)^2}{c^2\sigma}\right), 1\right)\right\} \cup \{R\}$ with $a, c \in F_{q^2}$, $c \neq 0$. Since any element of F_{q^2} can be written as $\frac{a^2+(c\sigma)^2}{c^2\sigma}$ for some elements $a, c \in F_{q^2}$, it follows that $O(R) = m^\perp$. This concludes the proof. $\square$

3.1. Case (a)

Let $\Sigma = \mathrm{Fix}(\tau)$ be a canonical subgeometry of $PG(3, q^2)$ such that $\Sigma \cap Q = \emptyset$ and let $Q \cap Q^\tau = \{l, l^\tau, m, m^\tau\}$, where $l, l^\tau \in \mathcal{R}$ and $m, m^\tau \in \mathcal{R}'$. Such a set of lines is called a quadrangle of Q. We will prove the following:

Theorem 3.5. *A canonical subgeometry Σ of type (a) is, up to the action of G,*

$$\Sigma = \{(x, y, dy^q, x^q) : x, y \in F_{q^2}\},$$

where $d \in F_{q^2} \setminus F_q$.

Proof. Since the group G acts transitively on the quadrangles of Q, we can suppose $l = \{(X_0, X_1, 0, 0) : X_0, X_1 \in F_{q^2}\}$, $l^\tau = \{(0, 0, X_2, X_3) : X_2, X_3 \in F_{q^2}\}$, $m = \{(X_0, 0, X_2, 0) : X_0, X_2 \in F_{q^2}\}$ and $m^\tau = \{(0, X_1, 0, X_3) : X_1, X_3 \in F_{q^2}\}$. Put $P = l \cap m = (1, 0, 0, 0)$ and $R = l \cap m^\tau = (0, 1, 0, 0)$; then $P^\tau = (0, 0, 0, 1)$, $R^\tau = (0, 0, 1, 0)$ and the lines $PP^\tau = \{(X_0, 0, 0, X_3) : X_0, X_3 \in F_{q^2}\}$ and $RR^\tau = \{(0, X_1, X_2, 0) : X_1, X_2 \in F_{q^2}\}$ are both fixed by τ and hence they contain $q + 1$ points fixed by τ. Therefore, by Proposition 3.4 (i), we can also suppose that the point $M = (1, 0, 0, 1)$ of PP^τ is fixed by τ, i.e. $M^\tau = M$. From these conditions we easily get that the matrix defining the semilinear collineation τ is

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & d^{-q} & 0 \\ 0 & d & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

with $d \in F_{q^2}^*$. Finally, since $Q \neq Q^\tau$, we also obtain $d \notin F_q$. Now, direct computation shows that the canonical subgeometry fixed by τ is $\Sigma = \mathrm{Fix}(\tau) = \{(x, y, dy^q, x^q) : x, y \in F_{q^2}\}$, $d \notin F_q$. This concludes the proof. □

3.2. Case (b)

Let $\Sigma = \mathrm{Fix}(\tau)$ be a canonical subgeometry of $PG(3, q^2)$ such that $\Sigma \cap Q = \emptyset$ and $Q \cap Q^\tau = \{C, C^\tau\}$, where C is a non-degenerate conic and $C \cap C^\tau = \{P, P^\tau\}$. We have the following:

Theorem 3.6. *A canonical subgeometry Σ of type (b) is, up to the action of G,*

$$\Sigma = \{(x, y, uy + vy^q, x^q) : x, y \in F_{q^2}\},$$

where u, v are elements of $F_{q^2}^*$ such that $x^{q+1} \neq uy^2 + vy^{q+1}$ for any $(x, y) \in F_{q^2} \times F_{q^2} \setminus \{(0, 0)\}$.

Proof. Since the group G acts transitively on the secants to Q, we can choose $P = (1, 0, 0, 0)$ and $P^\tau = (0, 0, 0, 1)$. Note that the line $r = PP^\tau$ contains $q + 1$ points fixed by τ. By Proposition 3.4 (i) we can suppose that the point $M = (1, 0, 0, 1)$ of PP^τ is fixed by τ. From $P^\tau = (0, 0, 0, 1)$, $P^{\tau^2} = P$ and $M^\tau = M$, we obtain that the matrix A defining τ has the form

$$A = \begin{pmatrix} 0 & a_{01} & a_{02} & 1 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ 1 & a_{31} & a_{32} & 0 \end{pmatrix}.$$

Let π_1 and π_2 be the planes of $PG(3, q^2)$ through the line r containing the conics C and C^τ respectively. Then $C = \pi_1 \cap Q = \{(s_1t^2, s_1t, t, 1) : t \in F_{q^2}\} \cup \{P\}$, for some $s_1 \in F^*_{q^2}$, and $C^\tau = \pi_2 \cap Q = \{(s_2t^2, s_2t, t, 1) : t \in F_{q^2}\} \cup \{P\}$, for some $s_2 \in F^*_{q^2}$, with $s_1 \neq s_2$. Since $Q \cap Q^\tau = \{C, C^\tau\}$, we have that $C^\tau, (C^\tau)^\tau \subseteq Q$, from which we get

$$a_{01}s_1^q + a_{02} = a_{01}s_2^q + a_{02} = 0$$

and

$$a_{31}s_1^q + a_{32} = a_{31}s_2^q + a_{32} = 0,$$

i.e., $a_{01} = a_{02} = a_{31} = a_{32} = 0$. Also, we obtain that s_1 and s_2 are the solutions of the equation

$$a_{11}^q a_{21}^q x^2 + (a_{22}^q a_{11}^q + a_{21}^q a_{12}^q - 1)x + a_{12}^q a_{22}^q = 0. \tag{2}$$

Since $s_1, s_2 \neq 0$, we have $a_{12}, a_{22} \neq 0$. From $\tau^2 = 1$, it follows $A^q A = \rho I$ with $\rho \neq 0$ and hence

$$a_{12}^q a_{11} + a_{12}a_{22}^q = 0, \qquad a_{21}^q a_{11} + a_{22}^q a_{21} = 0 \tag{3}$$

$$a_{21}^q a_{12} + a_{22}^{q+1} = 1, \qquad a_{11}^{q+1} + a_{12}^q a_{21} = 1. \tag{4}$$

Then the matrix A is

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

where $a_{11} = -\frac{a_{22}^q}{a_{12}^{q-1}}$ and $a_{21} = \frac{1-a_{22}^{q+1}}{a_{12}^q}$.

A direct computation shows that $\Sigma = \mathrm{Fix}(\tau) = \{(x, y, \frac{a_{22}}{a_{12}}y + \frac{1}{a_{12}^q}y^q, x^q) : x, y \in F_{q^2}\}$. By construction, $Q \cap Q^\tau$ contains two distinct conics; by requiring $Q \cap \Sigma = \emptyset$, we get that such conics are conjugate with respect to τ and we obtain $x^{q+1} \neq \frac{a_{22}}{a_{12}}y^2 + \frac{1}{a_{12}^q}y^{q+1}$, $\forall x, y \in F_{q^2}$. Put $u = \frac{a_{22}}{a_{12}}$ and $v = \frac{1}{a_{12}^q}$; then the theorem follows. □

3.3. *Case (c)*

Let $\Sigma = \mathrm{Fix}(\tau)$ be a canonical subgeometry of $\mathbb{P} = PG(3, q^2)$ such that $\Sigma \cap Q = \emptyset$ and $Q \cap Q^\tau = \emptyset$. We have the following result:

Theorem 3.7. *A canonical subgeometry Σ of type (c), up to the action of G, has the following forms:*

(i) *if q is odd*

$$\Sigma = \{(x - c_1x^q, y - c_1y^q a, y\sigma + \sigma a y^q c_1, x + c_1x^q) : x, y \in F_{q^2}\},$$

or

$$\Sigma = \{(x - a\sigma c_2y^q, y - c_2x^q, y\sigma + c_2\sigma x^q, x + a\sigma c_2y^q) : x, y \in F_{q^2}\},$$

where σ is a non-square in F_{q^2}, $a^2 = \sigma^{q-1}$ and c_1, c_2 are elements of F_{q^2} such that $c_1^{2(q+1)} \neq 1$ and $c_2^{2(q+1)} \neq \frac{1}{\sigma^{q+1}}$;

(ii) *if q is even*

$$\Sigma = \{(x + c(x^q + y^q b), y + cy^q, y\sigma + c(x^q + y^q(\sigma + b)), x + y + c(x^q + by^q)) : x, y \in F_{q^2}\},$$

where σ is an element of F_{q^2} such that $\mathrm{Tr}_{q^2/2}(\sigma) = 1$, b is a root in F_{q^2} of the polynomial $x^2 + x = \sigma^q + \sigma$ and c is an element of F_{q^2} such that $c^{q+1} \neq 1$.

Proof. Denote by τ^* the extension of τ to $\mathbb{P}^* = PG(3, q^4)$ and note that $\tau^{*2} : \mathbf{x} \in \mathbb{P}^* \to \mathbf{x}^{q^2} \in \mathbb{P}^*$. Hence τ^* has order 4 and $\mathrm{Fix}(\tau^{*2}) = \mathbb{P}$. Since $Q \cap Q^\tau = \emptyset$, then $Q^* \cap Q^{*\tau^*} = \{l, l^{\tau^*}, l^{\tau^{*2}}, l^{\tau^{*3}}\}$, where l and $l^{\tau^{*2}}$ belong to a ruling of Q^* and l^{τ^*} and $l^{\tau^{*3}}$ belong to the opposite one. Let m^* be the line joining the points $l \cap l^{\tau^*}$ and $l^{\tau^{*2}} \cap l^{\tau^{*3}}$. Note that the lines m^* and $m^{*\tau^*}$ are both fixed by τ^{*2}; this means that m^* and $m^{*\tau^*}$ determine two lines, say m and m^τ of $PG(3, q^2)$, respectively, both external to the quadric Q. Also note that $m^{*\perp^*} = m^{*\tau^*}$ (where $\perp^*$ is the polarity defined by Q^*) and hence $m^\perp = m^\tau$ (where $\perp$ is the polarity defined by Q). Since G acts transitively on the lines of $\mathbb{P}$ external to Q, we can fix the line m.

$\boxed{\textbf{q odd}}$ If q is odd, put $m = \{(X_0, X_1, \sigma X_1, X_0) : X_0, X_1 \in F_{q^2}\}$, where σ is a non-square in F_{q^2}. This implies that $l = l_{1,\phi}$, $l^{\tau^*} = l'_{1,\phi^{-1}}$, $l^{\tau^{*2}} = l_{1,-\phi}$ and $l^{\tau^{*3}} = l'_{1,-\phi^{-1}}$, where ϕ is an element of F_{q^4} such that $\phi^2 = \sigma$. Hence, if $N = l \cap l^\tau$, then

$$\begin{aligned} &N = (1, \phi^{-1}, \phi, 1), \quad N^{\tau^*} = (1, \phi^{-1}, -\phi, -1), \\ &N^{\tau^{*2}} = (1, -\phi^{-1}, -\phi, 1), \quad N^{\tau^{*3}} = (1, -\phi^{-1}, \phi, -1), \quad N^{\tau^{*4}} = N \end{aligned} \tag{*}$$

and $m^\tau = m^\perp = \{(X_0, X_1, -\sigma X_1, -X_0) : X_0, X_1 \in F_{q^2}\}$. Let M be the point of m with coordinates $(1, 0, 0, 1)$. By Proposition 3.4 (ii) the stabilizer of M in $\overline{G}_m$ has two orbits on the line $m^\perp$ and the points $R = (1, 0, 0, -1)$ and $T = (0, 1, -\sigma, 0)$ of $m^\perp$ belong to different orbits. This means that we may require that either $M^\tau = R$ or $M^\tau = T$.

Suppose $M^\tau = R$ and let $A_1 = (a_{ij})$, $i, j = 0, \ldots, 3$, be the matrix defining the semilinear collineation τ; from $M^\tau = R$ and $R^\tau = M$, we get

$$\begin{aligned} &a_{00} + a_{33} = 0, \qquad a_{03} + a_{30} = 0, \\ &a_{10} = a_{13} = a_{20} = a_{23} = 0. \end{aligned}$$

Combining the above relationships with Conditions (*) we obtain

$$\begin{aligned} &a_{01} = a_{02} = a_{31} = a_{32} = 0, \\ &a_{00} = \frac{a_{12}\sigma^q}{a}, \qquad a_{03} = \frac{a_{11}}{a}, \\ &a_{21} + \sigma^{q+1} a_{12} = 0, \qquad a_{22}\sigma^q + \sigma a_{11} = 0, \end{aligned}$$

where a is the element of F_{q^2} such that $\phi^q = a\phi$ (i.e. $a^2 = \sigma^{q-1}$). Finally, from $A_1^q A_1 = \rho I$, with $\rho \neq 0$, we get $\sigma a_{12}^q a_{11} \in F_q$. Put $\alpha = \sigma a_{12}^q a_{11}$, with $\alpha \in F_q$; since

$Q \neq Q^\tau$, we have $\alpha \neq 0$ and hence $a_{12}, a_{11} \neq 0$. So $a_{11} = \frac{\alpha}{\sigma a_{12}^q}$ and

$$A_1 = \begin{pmatrix} \frac{\sigma^q}{a} a_{12} & 0 & 0 & \frac{\alpha}{a\sigma a_{12}^q} \\ 0 & \frac{\alpha}{\sigma a_{12}^q} & a_{12} & 0 \\ 0 & -\sigma^{q+1} a_{12} & -\frac{\alpha}{\sigma^q a_{12}^q} & 0 \\ -\frac{\alpha}{a\sigma a_{12}^q} & 0 & 0 & -\frac{\sigma^q a_{12}}{a} \end{pmatrix}.$$

Also, since $\det A_1 \neq 0$, we have $\sigma^{q+1} a_{12}^{q+1} \neq \pm\alpha$. It is easy to verify that the canonical subgeometry Σ fixed by τ is

$$\Sigma = \{(x - c_1 x^q, y - c_1 y^q a, y\sigma + \sigma a y^q c_1, x + c_1 x^q) : x, y \in F_{q^2}\},$$

where c_1 is an element of F_{q^2} such that $c_1^{q+1} = \frac{\alpha + \sigma^{q+1} a_{12}^{q+1}}{\sigma^{q+1} a_{12}^{q+1} - \alpha}$. Note that $\alpha, a_{12} \neq 0$ implies $c_1^{q+1} \neq \pm 1$.

On the other hand, suppose that $M^\tau = T$ and let $A_2 = (b_{ij}), i, j = 0, \ldots, 3$ be the matrix associated with τ. By requiring $M^\tau = T$, $T^\tau = M$, we get the following conditions on the coefficients of A_2:

$$\begin{aligned} & b_{00} + b_{03} = 0, \qquad b_{30} + b_{33} = 0, \\ & b_{20} + b_{23} = -\sigma(b_{10} + b_{13}), \\ & b_{01} - \sigma^q b_{02} = b_{31} - \sigma^q b_{32}, \\ & b_{11} - \sigma^q b_{12} = 0, \qquad b_{21} - \sigma^q b_{22} = 0. \end{aligned}$$

Combining these relationships with Conditions (*) we obtain

$$\begin{aligned} & b_{00} = b_{03} = b_{30} = b_{33} = 0, \\ & b_{11} = b_{12} = b_{21} = b_{22} = 0, \\ & b_{01} = a\sigma b_{13}, \qquad b_{02} = \frac{a\sigma}{\sigma^q} b_{10}, \\ & b_{20} = -\sigma b_{13}, \qquad b_{23} = -\sigma b_{10}, \\ & b_{31} = -a\sigma b_{10}, \qquad b_{32} = -\frac{a\sigma}{\sigma^q} b_{13}, \end{aligned}$$

where a is the element of F_{q^2} such that $\phi^q = a\phi$. Finally, from $A_2^q A_2 = \rho I$, with $\rho \neq 0$, we obtain $b_{13} = \epsilon b_{10}$, where ϵ is an element of F_{q^2} such that $\epsilon^{q+1} = 1$ with $\epsilon \neq \pm 1$. Dividing by b_{10}, the matrix A_2 can be written as

$$A_2 = \begin{pmatrix} 0 & \sigma a \epsilon & \frac{a}{\sigma^q}\sigma & 0 \\ 1 & 0 & 0 & \epsilon \\ -\sigma\epsilon & 0 & 0 & -\sigma \\ 0 & -a\sigma & -\frac{\sigma a}{\sigma^q}\epsilon & 0 \end{pmatrix}.$$

It is now easy to verify that the subgeometry Σ fixed by τ is

$$\Sigma = \{(x - a\sigma c_2 y^q, y - c_2 x^q, y\sigma + c_2\sigma a y^q, x + a\sigma c_2 y^q) \colon x, y \in F_{q^2}\},$$

where c_2 is an element of $F_{q^2}^*$ such that $c_2^{q+1} = -\frac{1+\epsilon}{1-\epsilon}\frac{1}{a\sigma}$. If $c_2^{2(q+1)} = \frac{1}{\sigma^{q+1}}$, since $a^2\sigma^2 = \sigma^{q+1}$, we get $1 - \epsilon = \pm(1+\epsilon)$, which is not possible; so $c_2^{2(q+1)} \neq \frac{1}{\sigma^{q+1}}$.

$\boxed{\textbf{q even}}$ If q is even, let $m = \{(X_0, X_1, X_0 + \sigma X_1, X_0) \colon X_0, X_1 \in F_{q^2}\}$, where σ is an element of F_{q^2} such that the polynomial $x^2 + x + \sigma$ has no root in F_{q^2} (i.e., $\mathrm{Tr}_{q^2/2}(\sigma) = 1$) and let μ be an element of F_{q^4} such that $\mu^2 + \mu + \sigma = 0$. In this case we have $l = l_{1,\mu}$, $l^{\tau^*} = l'_{1,\frac{1}{1+\mu}}$, $l^{\tau^{*2}} = l_{1,1+\mu}$, $l^{\tau^{*3}} = l'_{1,\frac{1}{\mu}}$ and, if $N = l \cap l^{\tau^{*3}}$, then

$$\begin{aligned} &N = \left(1, \frac{1}{\mu}, \mu, 1\right), \quad N^{\tau^*} = \left(1, \frac{1}{1+\mu}, \mu, \frac{\mu}{1+\mu}\right), \\ &N^{\tau^{*2}} = \left(1, \frac{1}{\mu+1}, \mu+1, 1\right), \quad N^{\tau^{*3}} = \left(1, \frac{1}{\mu}, 1+\mu, \frac{1+\mu}{\mu}\right), \quad N^{\tau^{*4}} = N. \end{aligned} \tag{**}$$

Since $m^\tau = m^\perp$, we get $m^\tau = \{(X_0, X_1, \sigma X_1, X_0 + X_1) \colon X_0, X_1 \in F_{q^2}\}$. Let M be the point of m^τ with coordinates $(1, 0, 0, 1)$. By Proposition 3.4(ii), the stabilizer of M in $\overline{G}_m$ acts transitively on the points of $m^\perp$; hence we may require $M^\tau = R$, where $R = (1, 0, 1, 1) \in m$. Let $A = (a_{ij})$, $i, j = 0, \ldots, 3$, be the matrix defining the semilinear collineation τ. From $M^\tau = R$ and $R^\tau = M$ we get

$$\begin{aligned} &a_{00} + a_{03} = a_{20} + a_{23} = a_{30} + a_{33}, \\ &a_{10} = a_{13}, \quad a_{02} = a_{32}, \\ &a_{22} = a_{20} + a_{23}, \quad a_{12} = 0. \end{aligned}$$

Put $a_{00} + a_{03} = d$; by combining the above relationships with Conditions (**) we obtain

$$\begin{aligned} &a_{00} = a_{02}b^q, \qquad a_{01} = a_{02}(b^q + \sigma^q) + db^q, \qquad a_{03} = d + a_{02}b^q, \\ &a_{10} = a_{13} = a_{02}, \qquad a_{11} = a_{02} + d, \qquad a_{12} = 0, \\ &a_{20} = a_{02}\sigma + db^q, \qquad a_{21} = a_{02}\sigma + db^{q+1}, \qquad a_{22} = d, \qquad a_{23} = a_{02}\sigma + db, \\ &a_{30} = ba_{02} + d, \qquad a_{31} = a_{02}(b + \sigma^q) + bd, \qquad a_{32} = a_{02}, \qquad a_{33} = a_{02}b, \end{aligned}$$

where b is the element of F_{q^2} such that $\mu^q = b + \mu$. Finally, from $A^q A = \rho I$ with $\rho \neq 0$, we obtain $a_{02}d^q \in F_q^*$ and $a_{02} \neq d$; put $\alpha = a_{02}d^q$, with $\alpha \in F_q^*$. Hence, the matrix A is

$$A = \begin{pmatrix} a_{02}b^q & a_{02}(b^q + \sigma^q) + db^q & a_{02} & d + a_{02}b^q \\ a_{02} & a_{02} + d & 0 & a_{02} \\ a_{02}\sigma + db^q & a_{02}\sigma + db^{q+1} & d & a_{02}\sigma + db \\ ba_{02} + d & a_{02}(b + \sigma^q) + db & a_{02} & a_{02}b \end{pmatrix}.$$

It is now easy to verify that the subgeometry Σ fixed by τ has the following form:

$$\begin{aligned} \Sigma = \{&(x + c(x^q + y^q b), y + cy^q, y\sigma + c(x^q + y^q(\sigma + b)), \\ &x + y + c(x^q + by^q)) \colon x, y \in F_{q^2}\}, \end{aligned}$$

where c is an element of $F_{q^2}^*$ such that $c^{q+1} = \frac{d^{q+1}}{\alpha+d^{q+1}}$, in particular $c^{q+1} \neq 1$. This concludes the proof. □

4. The classification theorem

Let L be an F_q-linear set, with F_q as the maximal subfield of linearity, of rank 4 of $PG(3, q^2)$ disjoint from the hyperbolic quadric $Q^+(3, q^2)$ with equation $X_0X_3 - X_1X_2 = 0$. We can write

$$L = \{(h_0(x, y), h_1(x, y), h_2(x, y), h_3(x, y)) : x, y \in F_{q^2}\},$$

where $h_i : F_{q^2} \times F_{q^2} \to F_{q^2} (i = 0, \ldots, 3)$ are suitable linear functions over F_q. The linear set L is associated with a semifield spread $\mathcal{S}$ and with the spread set

$$\mathcal{C}_{\mathcal{S}} = \left\{ \begin{pmatrix} h_0(x, y) & h_1(x, y) \\ h_2(x, y) & h_3(x, y) \end{pmatrix} : x, y \in F_{q^2} \right\}.$$

The semifield plane $\pi(\mathcal{S})$ of order q^4 arising from $\mathcal{S}$ by the Andrè/Bruck–Bose construction is coordinatized by a (pre)semifield $(\mathcal{F}, +, \circ)$ with dimension 2 over its kernel N_l and dimension 4 over its center [9], where $\mathcal{F} = F_{q^2} \times F_{q^2}$, $N_l = F_{q^2} \times \{0\}$ and the (pre)semifield multiplication is

$$(x_0, x_1) \circ (x_2, x_3) = (x_0, x_1) \begin{pmatrix} h_0(x_2, x_3) & h_1(x_2, x_3) \\ h_2(x_2, x_3) & h_3(x_2, x_3) \end{pmatrix}.$$

The known classes of examples of (pre)semifields two-dimensional over their kernel and four-dimensional over their center are the following:

1. *Generalized Dickson semifields* $(\mathcal{F}, +, \circ)$ (see [9, p. 241]) with $\mathcal{F} = F_{q^2} \times F_{q^2}$, q odd, $N_l = F_{q^2} \times \{0\}$ and

$$(x_0, x_1) \circ (x_2, x_3) = (x_0, x_1) \begin{pmatrix} x_2 & x_3 \\ m x_3^q & x_2 \end{pmatrix}, \tag{5}$$

where m is a non-square in F_{q^2}. The linear set associated with $\mathcal{F}$ is $L = \{(x, y, my^q, x) : x, y \in F_{q^2}\}$ and it is easy to see that the point $P_0 = (1, 0, 0, 1)$ is the unique point of L of weight 2.

2. *Knuth semifields* $(\mathcal{F}, +, \circ)$ *of type* (17) (see [9, p. 241]), $\mathcal{F} = F_{q^2} \times F_{q^2}$, $N_l = F_{q^2} \times \{0\}$ and

$$(x_0, x_1) \circ (x_2, x_3) = (x_0, x_1) \begin{pmatrix} x_2 & x_3 \\ x_3^q f & x_2^q + x_3^q g \end{pmatrix}, \tag{6}$$

where f, g are elements of F_{q^2} such that $y^{q+1} + gy - f \neq 0 \,\forall y \in F_{q^2}$. The linear set associated with $\mathcal{F}$ is $\Sigma = \{(x, y, y^q f, x^q + y^q g) : x, y \in F_{q^2}\}$ and it is easy to check that

$\Sigma = \mathrm{Fix}(\tau)$ where τ is the semilinear collineation of $PG(3, q^2)$ of order 2:

$$\tau\colon (x_0, x_1, x_2, x_3) \to \begin{pmatrix} 0 & 0 & -g^q f^{-q} & 1 \\ 0 & 0 & f^{-q} & 0 \\ 0 & f & 0 & 0 \\ 1 & g & 0 & 0 \end{pmatrix} \begin{pmatrix} x_0^q \\ x_1^q \\ x_2^q \\ x_3^q \end{pmatrix}.$$

A straightforward computation shows that $Q \cap Q^\tau$ is a quadrangle; hence Σ is a canonical subgeometry of type (a).

3. *Knuth semifields* $(\mathcal{F}, +, \circ)$ *of type* (19) (see [9, p. 241]), $\mathcal{F} = F_{q^2} \times F_{q^2}$, $N_l = F_{q^2} \times \{0\}$ and

$$(x_0, x_1) \circ (x_2, x_3) = (x_0, x_1) \begin{pmatrix} x_2 & x_3 \\ x_3^q f & x_2^q + x_3 g \end{pmatrix} \tag{7}$$

where f, g are elements of F_{q^2} such that $y^{q+1} + gy - f \neq 0\, \forall y \in F_{q^2}$. The linear set associated with $\mathcal{F}$ is $\Sigma = \{(x, y, y^q f, x^q + yg) \colon x, y \in F_{q^2}\}$ and it is easy to check that $\Sigma = \mathrm{Fix}(\tau)$ where τ is defined as

$$\tau\colon (x_0, x_1, x_2, x_3) \to \begin{pmatrix} 0 & -g^q & 0 & 1 \\ 0 & 0 & f^{-q} & 0 \\ 0 & f & 0 & 0 \\ 1 & 0 & g f^{-q} & 0 \end{pmatrix} \begin{pmatrix} x_0^q \\ x_1^q \\ x_2^q \\ x_3^q \end{pmatrix}.$$

Also in this case it is easy to show that $Q \cap Q^\tau$ is a quadrangle; hence Σ is a canonical subgeometry of type (a).

If $g = 0$, then Knuth semifields of type (17) and (19) coincide and in this case the semifield plane coordinatized by them is called *Hughes–Kleinfeld semifield* plane.

4. *Semifield planes lifted from Desarguesian planes*

It is possible to construct a class of semifield planes of order q^4, with kernel F_{q^2}, starting from semifield planes of order q^2 with kernel containing F_q. This construction method is called *lifting* and it has been introduced by Hiramine, Matsumoto and Oyama in [11], for q odd, and it has been generalized by Johnson in [13] to any q (for further details on lifting see e.g. [3]).

Let π be any semifield plane of order q^2 with kernel containing F_q. Then there is a spread set $\mathcal{C}$ associated with π that has the following form:

$$\mathcal{C} = \left\{ \begin{pmatrix} u & t \\ f(u,t) & g(u,t) \end{pmatrix} : u, t \in F_q \right\},$$

where f and g are additive functions. Let F_{q^2} have basis $\{\theta, 1\}$ over F_q, where $\theta^2 = \theta\alpha + \beta$. Since f and g are additive, the lifted plane π^L is defined by the spread set

$$\mathcal{C}^L = \left\{ \begin{pmatrix} l & \theta t + u \\ -\theta g(u,t) + f(u,t) + \alpha g(u,t) & l^q \end{pmatrix} : u, t \in F_q, l \in F_{q^2} \right\}.$$

Let $K(\pi)$ be the center of π and let $\dim_{K(\pi)}(\mathcal{C}) = 2h$ (i.e., $K(\pi) = F_{q'}$ and $q = q'^h$). Since $K(\pi)$ is the maximal subfield of F_q with respect to which $\mathcal{C}$ is a $K(\pi)$-vector space,

$K(\pi)$ is the maximal subfield of F_q such that $f(\lambda u, \lambda t) = \lambda f(u,t)$ and $g(\lambda u, \lambda t) = \lambda g(u,t)$ for any $\lambda \in K(\pi)$ and for any $u, t \in F_q$. Note that $K(\pi) = F_q$ if and only if f and g are linear functions over F_q, i.e. if and only if the plane π is Desarguesian.

Theorem 4.1. *Let π be a semifield plane of order q^2 with kernel containing F_q and center $K(\pi) = F_{q'}$ $(q = q'^h)$. Then the lifted plane π^L has center $K(\pi^L) = K(\pi)$. Hence $\dim_{K(\pi^L)}\mathcal{C}^L = 2\dim_{K(\pi)}\mathcal{C} = 4h$.*

Proof. Recall that $K(\pi^L)$ is the maximal subfield of F_{q^2} such that $\lambda X \in \mathcal{C}^L$ for any $\lambda \in K(\pi^L)$ and for any $X \in \mathcal{C}^L$. If $X \in \mathcal{C}^L$ then $\lambda X \in \mathcal{C}^L$ if and only if there exist $t', u' \in F_q$ and $l' \in F_{q^2}$ such that

$$\lambda \begin{pmatrix} l & \theta t + u \\ -\theta g(u,t) + f(u,t) + \alpha g(u,t) & l^q \end{pmatrix} = \begin{pmatrix} l' & \theta t' + u' \\ -\theta g(u',t') + f(u',t') + \alpha g(u',t') & l'^q \end{pmatrix}.$$

This implies that $\lambda l^q = l'^q$ and $\lambda l = l'$ for any l, i.e. $\lambda \in F_q$. Also, if $\lambda \in F_q$, from $\lambda(\theta t + u) = \theta t' + u'$ we get $t' = \lambda t$ and $u' = \lambda u$ and from

$$\lambda(-\theta g(u,t) + f(u,t) + \alpha g(u,t)) = -\theta g(\lambda u, \lambda t) + f(\lambda u, \lambda t) + \alpha g(\lambda u, \lambda t)$$

we get $f(\lambda u, \lambda t) = \lambda f(u,t)$, $g(\lambda u, \lambda t) = \lambda g(u,t)$ for any $u, t \in F_q$. Therefore $K(\pi^L) = K(\pi)$. $\square$

Corollary 4.1. *A lifted plane π^L of order q^4 has center F_q if and only if π is a Desarguesian plane.*

Proof. By Theorem 4.1, π^L has center F_q if and only if $K(\pi) = F_q$, i.e. if and only if π is Desarguesian. $\square$

Now, lifting a Desarguesian plane of order q^2 defined by a spread set $\mathcal{C}$ with $f(u,t)$ and $g(u,t)$ linear over F_q, we get a spread set

$$\mathcal{C}^L = \left\{ \begin{pmatrix} l & v \\ h(v) & l^q \end{pmatrix} : l, v \in F_{q^2} \right\},$$

where $v = \theta t + u$ and $h(v) = -\theta g(u,t) + f(u,t) + \alpha g(u,t)$. Since $h(\lambda v) = \lambda h(v)$ for any λ in F_q and $v \in F_{q^2}$, we can write $h(v) = f_0 v + f_1 v^q$, with $f_0, f_1 \in F_{q^2}$, i.e.

$$\mathcal{C}^L = \left\{ \begin{pmatrix} l & v \\ f_0 v + f_1 v^q & l^q \end{pmatrix} : l, v \in F_{q^2} \right\}.$$

Hence, the semifields $(\mathcal{F}, +, \circ)$, $\mathcal{F} = F_{q^2} \times F_{q^2}$, which coordinatize the planes obtained by lifting Desarguesian planes have multiplication

$$(x_0, x_1) \circ (x_2, x_3) = (x_0, x_1) \begin{pmatrix} x_2 & x_3 \\ f_0 x_3 + f_1 x_3^q & x_2^q \end{pmatrix} \tag{8}$$

where f_0, f_1 are elements of F_{q^2} such that $x^{q+1} \neq f_0 y^2 + f_1 y^{q+1}$ $\forall (x,y) \in (F_{q^2} \times F_{q^2}) \setminus \{(0,0)\}$. If $f_0 = 0$ and $f_1 \notin F_q$ then it is clear that we get the semifield multiplication of a Hughes–Kleinfeld semifield plane. If $f_1 = 0$, q is odd and f_0 is non-square in F_{q^2}, it is easy to see that we get the semifield multiplication of a plane isomorphic to a generalized Dickson semifield plane. So, suppose $f_0, f_1 \neq 0$. In this case the linear set associated with $\mathcal{F}$ is $\Sigma = \{(x, y, f_0 y + f_1 y^q, x^q): x, y \in F_{q^2}\}$ and it is easy to check that $\Sigma = \mathrm{Fix}(\tau)$, where τ is the semilinear collineation of $PG(3, q^2)$ of order 2 defined as follows:

$$\tau: (x_0, x_1, x_2, x_3) \to \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -f_0^q f_1^{-q} & f_1^{-q} & 0 \\ 0 & \dfrac{f_1^{q+1} - f_0^{q+1}}{f_1^q} & f_0 f_1^{-q} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_0^q \\ x_1^q \\ x_2^q \\ x_3^q \end{pmatrix}.$$

A straightforward computation shows that $Q \cap Q^\tau$ consists of two non-degenerate conjugate conics intersecting in conjugate points; hence Σ is a canonical subgeometry of type (b). Hence:

Proposition 4.2. *A semifield plane of order q^4 lifted from a Desarguesian plane is either a generalized Dickson semifield plane or a Hughes–Kleinfeld semifield plane or it is a plane of type (b).* □

For q odd, the lifted planes of type (b) are included in the class of planes studied by Cordero and Figueroa in [7, Theorem 3.1].

5. *Generalized twisted fields* $\mathcal{F}_c = (\mathcal{F}, +, \circ)$ (see [1]), $\mathcal{F} = F_{q^4}$, with dimension 2 over their left nucleus and 4 over their center. The presemifield multiplication is

$$x \circ y = xy - c x^{q^2} y^q \tag{9}$$

with $c \in F_{q^4}$ such that $N_{q^4/q}(c) = c^{q^3+q^2+q+1} \neq 1$.

Let $\mathcal{F}_c$ and $\mathcal{F}_d$ be two generalized twisted fields defined by c and d respectively. If $N_{q^4/q}(c) = N_{q^4/q}(d)$, then the presemifields $\mathcal{F}_c$ and $\mathcal{F}_d$ are *isotopic*, i.e., the semifields planes coordinated by $\mathcal{F}_c$ and $\mathcal{F}_d$ are isomorphic (see [2, Theorem 1]). Let $\{1, \lambda\}$ be a basis of F_{q^4} over F_{q^2}.

5.1. Let q be odd and take $\lambda \in F_{q^4}$ such that $\lambda^2 = \sigma$, where σ is a non-square in F_{q^2}. If $N_{q^4/q}(c)$ is a square in F_q, then there exists an element $c_1 \in F_{q^2}$ such that $N_{q^4/q}(c) = N_{q^4/q}(c_1)$; hence $\mathcal{F}_c$ is isotopic to $\mathcal{F}_{c_1}$ and the (pre)semifield multiplication (9) with $c = c_1$ can be rewritten in coordinate terms with respect to the basis $\{1, \lambda\}$ as

$$(x_0, x_1) \circ (x_2, x_3) = (x_0, x_1) \begin{pmatrix} x_2 - c_1 x_2^q & x_3 - c_1 x_3^q a \\ x_3\sigma + \sigma a c_1 x_3^q & x_2 + c_1 x_2^q \end{pmatrix} \tag{10}$$

where a is the element of F_{q^2} such that $\lambda^q = a\lambda$ (in particular, $a^2 = \sigma^{q-1}$). Since $c^{q^3+q^2+q+1} \neq 1$, we get $c_1^{2(q+1)} \neq 1$. The linear set associated with $\mathcal{F}_{c_1}$ is $\Sigma = \{(x - c_1 x^q, y - c_1 y^q a, \sigma(y + a c_1 y^q), x + c_1 x^q): x, y \in F_{q^2}\}$ and it is not difficult to check that $\Sigma = \mathrm{Fix}(\tau)$ where τ is the semilinear collineation of $PG(3, q^2)$ of order 2 defined as follows:

$$\tau\colon (x_0,x_1,x_2,x_3)\to \begin{pmatrix} -\dfrac{1+c_1^{q+1}}{2c_1^q} & 0 & 0 & \dfrac{1-c_1^{q+1}}{2c_1^q} \\ 0 & -\dfrac{1+(ac_1)^{q+1}}{2a^q c_1^q} & \dfrac{1-(ac_1)^{q+1}}{2a^q\sigma^q c_1^q} & 0 \\ 0 & \dfrac{-\sigma+\sigma(ac_1)^{q+1}}{2a^q c_1^q} & \dfrac{\sigma+\sigma(ac_1)^{q+1}}{2a^q\sigma^q c_1^q} & 0 \\ \dfrac{-1+c_1^{q+1}}{2c_1^q} & 0 & 0 & \dfrac{1+c_1^{q+1}}{2c_1^q} \end{pmatrix} \begin{pmatrix} x_0^q \\ x_1^q \\ x_2^q \\ x_3^q \end{pmatrix}.$$

A straightforward computation shows that $Q\cap Q^\tau=\emptyset$; hence Σ is a canonical subgeometry of type (c).

If $N_{q^4/q}(c)$ is a non-square in F_q, then it is easy to see that there exists an element $c_2\in F_{q^2}$ such that $N_{q^4/q}(c)=N_{q^4/q}(\lambda c_2)$. Hence the (pre)semifield $\mathcal{F}_c$ is isotopic to the (pre)semifield $\mathcal{F}_{c_2\lambda}$ and the (pre)semifield multiplication (9) with $c=c_2\lambda$ can be rewritten in coordinate terms as

$$(x_0,x_1)\circ(x_2,x_3)=(x_0,x_1)\begin{pmatrix} x_2-c_2\sigma a x_3^q & x_3-c_2x_2^q \\ x_3\sigma+c_2\sigma x_2^q & x_2+a\sigma c_2 x_3^q \end{pmatrix} \tag{11}$$

where a is the element of F_{q^2} such that $\lambda^q=a\lambda$ (in particular, $a^2=\sigma^{q-1}$). Since $c^{q^3+q^2+q+1}\neq 1$, we get $c_2^{2(q+1)}\sigma^{(q+1)}\neq 1$. The linear set associated with $\mathcal{F}_{\lambda c_2}$ is $\Sigma=\{(x-c_2\sigma a y^q, y-c_2x^q, \sigma(y+c_2x^q), x+a\sigma c_2 y^q)\colon x,y\in F_{q^2}\}$ and it is not difficult to check that $\Sigma=\mathrm{Fix}(\tau)$ where τ is

$$\tau\colon (x_0,x_1,x_2,x_3)\to \begin{pmatrix} 0 & -\dfrac{1+c_2^{q+1}a\sigma}{2c_2^q} & \dfrac{1-c_2^{q+1}a\sigma}{2c_2^q\sigma^q} & 0 \\ -\dfrac{1+a^q\sigma^q c_2^{q+1}}{2a^q c_2^q\sigma^q} & 0 & 0 & \dfrac{1-a^q\sigma^q c_2^{q+1}}{2a^q\sigma^q c_2^q} \\ \dfrac{-\sigma+a^q(\sigma c_2)^{q+1}}{2a^q\sigma^q c_2^q} & 0 & 0 & \dfrac{\sigma+a^q(c_2\sigma)^{q+1}}{2a^q\sigma^q c_2^q} \\ 0 & \dfrac{-1+c_2^{q+1}a\sigma}{2c_2^q} & \dfrac{1+c_2^{q+1}a\sigma}{2c_2^q\sigma^q} & 0 \end{pmatrix} \begin{pmatrix} x_0^q \\ x_1^q \\ x_2^q \\ x_3^q \end{pmatrix}.$$

A straightforward computation shows that $Q\cap Q^\tau=\emptyset$; hence Σ is a canonical subgeometry of type (c).

5.2. Let q be even and take $\lambda\in F_{q^4}$ such that $\lambda^2+\lambda+\sigma=0$, $\sigma\in F_{q^2}$ and $\mathrm{Tr}_{q^2/2}(\sigma)=1$. If $\bar{c}\in F_{q^4}$ such that $N_{q^4/q}(\bar{c})\neq 1$, then it is easy to see that there exists an element $c\in F_{q^2}$ such that $N_{q^4/q}(\bar{c})=N_{q^4/q}(c)$. This means that we can suppose $c\in F_{q^2}$ and in this case multiplication (9) can be rewritten in coordinate terms with respect to the basis $\{1,\lambda\}$ as

$$\begin{aligned}(x_0,x_1)&\circ(x_2,x_3)\\ &=(x_0,x_1)\begin{pmatrix} x_2+cx_2^q+cx_3^q b & cx_3^q+x_3 \\ cx_2^q+cbx_3^q+x_3^q\sigma c+x_3\sigma & x_2+cx_2^q+x_3+cbx_3^q \end{pmatrix}\end{aligned} \tag{12}$$

where $b=\lambda^q+\lambda$ (in particular b is a root in F_{q^2} of the polynomial $x^2+x=\sigma^q+\sigma$). Since $c^{q^3+q^2+q+1}\neq 1$, we get $c^{2(q+1)}\neq 1$. The linear set Σ associated with $\mathcal{F}_c$ is

$\Sigma = \{(x + c(x^q + by^q), y + cy^q, y\sigma + c(x^q + y^q(b+\sigma)), x + y + c(x^q + y^q b)) : x, y \in F_{q^2}\}$ and it is easy to check that $\Sigma = \mathrm{Fix}(\tau)$ where τ is the semilinear collineation of $PG(3, q^2)$ of order 2 defined as

$$\tau : (x_0, x_1, x_2, x_3) \to \begin{pmatrix} \frac{b^q(1+c^{q+1})}{c^q} & \frac{b^q + (1+c^{q+1})\sigma^q}{c^q} & \frac{1+c^{q+1}}{c^q} & \frac{bc^{q+1}+b^q}{c^q} \\ \frac{1+c^{q+1}}{c^q} & \frac{1}{c^q} & 0 & \frac{1+c^{q+1}}{c^q} \\ \frac{c^{q+1}(b^q+\sigma)+\sigma}{c^q} & \frac{\sigma+\sigma^q c^{q+1}}{c^q} & c & \frac{c^{q+1}(b+\sigma)+\sigma}{c^q} \\ \frac{c^{q+1}b^q+b}{c^q} & \frac{b+\sigma^q(1+c^{q+1})}{c^q} & \frac{1+c^{q+1}}{c^q} & \frac{1+bc^{q+1}}{c^q} \end{pmatrix} \begin{pmatrix} x_0^q \\ x_1^q \\ x_2^q \\ x_3^q \end{pmatrix}.$$

A straightforward computation shows that $Q \cap Q^\tau = \emptyset$; hence Σ is a canonical subgeometry of type (c).

We will say that a semifield plane π of order q^4, with kernel F_{q^2} and center F_q, is *of type (a)* (respectively *of type (b)* or *of type (c)*) if the canonical subgeometry (see Section 3) associated with π is of type (a) (respectively, of type (b) or of type (c)). By the above results, the following holds:

Theorem 4.3. *Let π be a semifield plane of order q^4 with kernel F_{q^2} and center F_q. Then:*

(i) *If π is a generalized Dickson semifield plane then it has a point of weight 2.*
(ii) *If π is a Knuth semifield plane of type* (17) *or* (19)*, then it is a semifield plane of type (a).*
(iii) *If π is a semifield plane lifted from a Desarguesian plane with semifield multiplication* (8) *and $f_0, f_1 \neq 0$, then it is a semifield plane of type (b).*
(iv) *If π is a generalized twisted field plane, then it is a semifield plane of type (c).* □

Furthermore, we can extend to any q the classification result on lifted semifield planes from Desarguesian planes of order q^2 (q odd) obtained by Cordero and Figueroa in [7, Theorem 3.1].

Theorem 4.4. *The semifield planes of order q^4 obtained by lifting Desarguesian planes are either generalized Dickson semifield planes, or Hughes–Kleinfeld semifield planes or semifield planes not isomorphic to any plane included in the classes* 1, 2, 3 *and* 5 *listed at the beginning of Section* 4*.*

Proof. By Proposition 4.2, a plane lifted from a Desarguesian semifield plane of order q^2 is either a generalized Dickson semifield plane, or a Hughes–Kleinfeld semifield plane, or it is a plane of type (b). By Theorem 2.1 and Remark 3.3, a plane of type (b) is not isomorphic to any plane of different type. By Theorem 4.3, planes in class 1 have a point of weight 2, planes in classes 2 and 3 are of type (a) and planes in class 5 are of type (c); hence planes of type (b) are not isomorphic to any plane included in the classes 1, 2, 3 and 5. □

The following theorem classifies semifield planes of order q^4 with kernel F_{q^2} and center F_q.

Theorem 4.5. *Let π be a semifield plane of order q^4 with kernel F_{q^2} and center F_q.*

(i) *If q is odd, then π belongs to one of the following classes: generalized Dickson semifield planes, Hughes–Kleinfeld semifield planes, semifield planes lifted from Desarguesian planes of Cordero–Figueroa type or generalized twisted field planes.*

(ii) *If q is even, then π either belongs to one of the following classes: Hughes–Kleinfeld semifield planes, generalized twisted field planes, or it belongs to the class of lifted planes (from Desarguesian planes) of type (b).*

Proof. Let π be a semifield of order q^4 two-dimensional over its kernel and four-dimensional over its center. The plane π is associated with an F_q-linear set L of $PG(3, q^2)$ of rank 4, with F_q as the maximal subfield of linearity, disjoint from the hyperbolic quadric with equation $X_0X_3 - X_1X_2 = 0$. Recall that L contains at most one point of weight 2. If this is the case, by Proposition 3.1 (up to the action of G) L has the following form:

$$L = \{(x, y, my^q, x) \colon (x, y) \in (F_{q^2} \times F_{q^2}) \setminus (0, 0)\},$$

where m is a non-square in F_{q^2}. This means that the semifield associated with L has multiplication defined by

$$(x_0, x_1) \circ (x_2, x_3) = (x_0, x_1) \begin{pmatrix} x_2 & x_3 \\ mx_3^q & x_2 \end{pmatrix};$$

hence by Theorem 2.1 the plane π is isomorphic to a generalized Dickson semifield plane.

Now, suppose that L has no point of weight two. Then $L = \Sigma$ is a canonical subgeometry of $PG(3, q^2)$. If Σ is of type (a), by Theorem 3.5 we can write

$$\Sigma = \{(x, y, dy^q, x^q) \colon x, y \in F_{q^2}\},$$

where $d \in F_{q^2} \setminus F_q$. Hence the semifield associated with this subgeometry has multiplication given by

$$(x_0, x_1) \circ (x_2, x_3) = (x_0, x_1) \begin{pmatrix} x_2 & x_3 \\ dx_3^q & x_2^q \end{pmatrix};$$

i.e., we obtain the semifield multiplication of Knuth type (17) and Knuth type (19) semifields with $g = 0$. Therefore, by Theorem 2.1, it follows that a semifield plane of type (a) is isomorphic to a Hughes–Kleinfeld semifield plane.

Suppose that Σ is a canonical subgeometry of type (b). Then, by Theorem 3.6, Σ can be written as

$$\Sigma = \{(x, y, uy + vy^q, x^q) \colon x, y \in F_{q^2}\},$$

where u, v are elements of $F^*_{q^2}$ such that $x^{q+1} \neq y(uy + vy^q)$ for any $(x, y) \in F_{q^2} \times F_{q^2} \setminus \{(0, 0)\}$.
This implies that the semifield multiplication is

$$(x_0, x_1) \circ (x_2, x_3) = (x_0, x_1) \begin{pmatrix} x_2 & x_3 \\ ux_3 + vx_3^q & x_2^q \end{pmatrix}. \tag{13}$$

By putting $f_0 = u$ and $f_1 = v$ we obtain the multiplication (8); this proves that a semifield plane π of type (b) is isomorphic to a semifield plane lifted from a Desarguesian plane.

Suppose that Σ is a canonical subgeometry of type (c). If q is odd, by Theorem 3.7, we have

$$\Sigma = \{(x - c_1 x^q, y - c_1 y^q a, y\sigma + \sigma a y^q c_1, x + c_1 x^q) \colon x, y \in F_{q^2}\}, \tag{1a}$$

or

$$\Sigma = \{(x - a\sigma c_2 y^q, y - c_2 x^q, y\sigma + c_2\sigma x^q, x + a\sigma c_2 y^q) \colon x, y \in F_{q^2}\}, \tag{1b}$$

where σ is a non-square in F_{q^2}, $a^2 = \sigma^{q-1}$, c_1 is an element of $F^*_{q^2}$ such that $c_1^{2(q+1)} \neq 1$ and c_2 is an element of $F^*_{q^2}$ such that $c_2^{2(q+1)} \neq \frac{1}{\sigma^{q+1}}$.
The semifield multiplication arising from (1a) is

$$(x_0, x_1) \circ (x_2, x_3) = (x_0, x_1) \begin{pmatrix} x_2 - c_1 x_2^q & x_3 - c_1 a x_3^q \\ x_3\sigma + \sigma a c_1 x_3^q & x_2 + c_1 x_2^q \end{pmatrix}, \tag{14}$$

which gives the multiplication (10) of a generalized twisted field $\mathcal{F}_c$ with $c \in F_{q^2}$.
The semifield multiplication arising from (1b) is

$$(x_0, x_1) \circ (x_2, x_3) = (x_0, x_1) \begin{pmatrix} x_2 - a\sigma c_2 x_3^q & x_3 - c_2 x_2^q \\ x_3\sigma + \sigma c_2 x_2^q & x_2 + a\sigma c_2 x_3^q \end{pmatrix}, \tag{15}$$

which gives the multiplication (11) of a generalized twisted field $\mathcal{F}_{\lambda c}$ with $c \in F_{q^2}$. By Theorem 2.1, it follows that when q is odd, a semifield plane of type (c) is isomorphic to a generalized twisted field plane.

Finally, if q is even, by Theorem 3.7, Σ can be written in the following way:

$$\begin{aligned}\Sigma = \{&(x + c(x^q + y^q b), y + cy^q, y\sigma + c(x^q + (\sigma + b)y^q),\\ &x + y + c(x^q + by^q)) \colon x, y \in F_{q^2}\}\end{aligned}$$

where σ is an element of F_{q^2} such that $\mathrm{Tr}_{q^2/2}(\sigma) = 1$, b is a root in F_{q^2} of the polynomial $x^2 + x = \sigma^q + \sigma$ and c is an element of F_{q^2} such that $c^{q+1} \neq 1$. The semifield multiplication arising from Σ is

$$\begin{aligned}&(x_0, x_1) \circ (x_2, x_3)\\ &\quad = (x_0, x_1) \begin{pmatrix} x_2 + c(x_2^q + x_3^q b) & x_3 + c x_3^q \\ x_3\sigma + c(x_2^q + (\sigma + b)x_3^q) & x_2 + x_3 + c(x_2^q + b x_3^q) \end{pmatrix}.\end{aligned} \tag{16}$$

Comparing with multiplication (12) makes it clear that (16) defines the generalized twisted field $\mathcal{F}_c$ with $c \in F_{q^2}$. This proves that also when q is even, a semifield plane of type (c) is isomorphic to a generalized twisted field plane. □

As a direct consequence of the previous theorem we get:

Theorem 4.6. *A semifield plane of order p^4, p prime, with kernel containing F_{p^2} is isomorphic to one of the following planes: Desarguesian plane, generalized Dickson semifield planes, Hughes–Kleinfeld semifield planes, generalized twisted field planes, semifield planes lifted from Desarguesian planes coordinatized by a semifield with multiplication* (8) *where* $f_0, f_1 \neq 0$.

Proof. Let π be a semifield plane of order p^4, p prime, with kernel containing F_{p^2}. If π is not a Desarguesian plane, then π has kernel F_{p^2} and center F_p. Now the result follows from Theorem 4.5. □

4.1. *Semifield planes of type (b), q even*

In this section we exhibit examples of planes of type (b), q even. Let π be a semifield plane of type (b). By (13), the multiplication of the associated semifield is

$$(x_0, x_1) \circ (x_2, x_3) = (x_0, x_1) \begin{pmatrix} x_2 & x_3 \\ ux_3 + vx_3^q & x_2^q \end{pmatrix} \tag{17}$$

where u, v are elements of $F_{q^2}^*$ satisfying the condition: $x^{q+1} \neq uy^2 + vy^{q+1}$ $\forall (x, y) \in F_{q^2} \times F_{q^2} \setminus \{(0, 0)\}$, which is equivalent to

$$uy^2 + vy^{q+1} \notin F_q \ \forall y \in F_{q^2}^*. \tag{18}$$

Let q be even. Semifield planes of type (b) exist if and only if there exist elements u, v in $F_{q^2}^*$ satisfying Condition (18). First note that if v is an element of F_q, then Condition (18) is not fulfilled for all $u \in F_{q^2}$. Hence, fix $v \notin F_q$ and write $v^2 = e + fv$, with $e, f \in F_q^*$ and $\mathrm{Tr}_{q/2}(\frac{e}{f^2}) = 1$ (note that $e = v^{q+1}$ and $f = v + v^q$).

Since $\{1, v\}$ is an F_q-basis of F_{q^2}, we can write $u = l + mv$, $y = y_0 + y_1 v$ where $l, m, y_0, y_1 \in F_q$ and Condition (18), in terms of coordinates, is equivalent to

$$y_0^2(m + 1) + fy_0y_1 + (lf + me + f^2m + e)y_1^2 \neq 0. \tag{19}$$

$\forall (y_0, y_1) \in F_{q^2} \times F_{q^2} \setminus \{(0, 0)\}$.

Such condition is satisfied if and only if

$$\mathrm{Tr}_{q/2}\left(\frac{(lf + me + f^2m + e)(m + 1)}{f^2}\right) = 1. \tag{20}$$

Now, if we choose $u = d(e + fv)$, where d is an element of F_q^* such that $\mathrm{Tr}_{q/2}(\frac{de}{f}) = 0$, then $l = de$, $m = df$ and (20) becomes

$$\mathrm{Tr}_{q/2}\left(\frac{(f^3d + e)(fd + 1)}{f^2}\right) = \mathrm{Tr}_{q/2}(d^2 f^2 + df) + \mathrm{Tr}_{q/2}\left(\frac{e}{f^2}\right) + \mathrm{Tr}_{q/2}\left(\frac{de}{f}\right) = 1.$$

This means that if we choose $v \notin F_q$ and $u = d(e + fv)$ (where $v^2 = e + fv$ and $\mathrm{Tr}_{q/2}(\frac{de}{f}) = 0$), Condition (18) is satisfied, i.e., we get examples of semifield planes of type (b), with q even.

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